Do six of the nine questions. Of these at least one should be from each section. If you work more than the required number of problems, make sure that you clearly mark which problems you want to have counted. If you have doubts about the wording of a problem or about what results may be assumed without proof, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial. We write $[n]$ for the set $\{1, 2, \ldots, n\}$.

Section A

Question 1. a) A *derangement* is a permutation that has no fixed points. Denote by D_n the number of derangements of the set $\{1, 2, \ldots, n\}$. Prove the following recurrence

$$D_n = (n-1)(D_{n-1} + D_{n-2}),$$

for any $n \geq 3$. *Note: If you would like to use any explicit formula for D_n, you need to prove it.*

b) Prove that

$$D_n = nD_{n-1} + (-1)^n,$$

for any $n \geq 2$.

Question 2. A circle is divided into p equal arcs, where p is a prime number. Each arc is colored by one of n different colors. How many nonequivalent colorings are there? Two colorings are equivalent if one of them can be obtained from the other one by a rotation of the circle about its center.

Question 3.

a) For integers $n \geq 0$ and $k > 0$, let a_n be the number of integer solutions of the equation

$$x_1 + x_2 + \cdots + x_k = n,$$

where $x_1 \geq x_2 \geq \cdots \geq x_k \geq 0$. Find the ordinary generating function for the sequence $(a_n)_{n\geq 0}$.

b) For integers $n \geq 0$ and $k > 0$, let b_n be the number of integer solutions of the equation

$$x_1 + x_2 + \cdots + x_k = n,$$

where $x_1 \geq x_2 \geq \cdots \geq x_k \geq 1$. Find the ordinary generating function for the sequence $(b_n)_{n\geq 0}$.

Section B

Question 4.

a) State (without proving) Sperner's Lemma concerning the maximum size of an antichain in $\mathcal{P}(n)$.

b) Let $a_1, a_2, \ldots, a_n$ be real numbers of absolute value at least one. For any open unit interval I, prove that there are at most $\binom{n}{\lfloor n/2 \rfloor}$ vectors $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \{-1, 1\}^n$ (i.e., ε is a vector whose entries are 1 or -1) such that $\sum_{i=1}^n \varepsilon_i a_i \in I$.

Question 5.

a) State (but you need not prove) Dilworth's theorem concerning chain decompositions of posets.

b) Let $I_1, I_2, \ldots, I_{10}$ be intervals on the real number line. Suppose that no four of the intervals are pairwise disjoint. Show that there must be four intervals that share a common point.
Hint: Define an appropriate partial order on the intervals.

Question 6. Given $k \leq n \leq q \in \mathbb{N}$, where q is a power of a prime, and a collection of distinct elements $\alpha_1, \ldots, \alpha_n \in \mathbb{F}_q$, we define $f : \mathbb{F}_q^k \to \mathbb{F}_q^n$ by

$$f(c_0, c_1, \ldots, c_{k-1}) = \left(c_0 + c_1\alpha_i + c_2\alpha_i^2 + \cdots + c_{k-1}\alpha_i^{k-1} \right)_{i \in [n]}.$$

a) Show that f is injective and it defines a linear $[n, k, d]$-code (i.e. a linear code of length n, dimension k and minimum distance d). Determine the value of d.
b) Determine the maximum number $t = t(k, n)$ of erasures that can be fixed. More precisely, suppose that we send a codeword $(w_1, \ldots, w_n)$ across a q-ary erasure channel, and ℓ symbols are erased (i.e. each erased w_i is replaced by an ϵ symbol).
   i. For $\ell \leq t$, give a method to successfully recover the transmitted codeword.
   ii. For $\ell \geq t + 1$, show that there are instances in which this is not possible.

Section C

Question 7. The parts of this question are unrelated.
a) Let G be a graph containing k edge-disjoint spanning trees, and let $e_1, \ldots, e_k$ be distinct edges in G. Prove that G has edge-disjoint spanning trees $T_1, \ldots, T_k$ such that $e_i \in T_i$ for $1 \leq i \leq k$.
b) Let x and y be vertices in a 3-connected graph G. Prove that G contains an x, y-path P such that $G - V(P)$ is connected.

Question 8. Suppose that G is a 3-regular graph with the property that all its edge-colorings with $\chi'(G)$ colors are identical, up to a permutation of the colors. Show that $\chi'(G) = 3$ and that G has exactly 3 Hamilton cycles.

Question 9. Let $n \geq 6$ be even, and let $F = \{e_1, \ldots, e_{n-1}\}$ be a collection of $n - 1$ different edges of K_n that are not all incident with one same vertex. Show that $K_n - F$ has a perfect matching.